PHYSICAL REVIEW E **69**, 025601 (2004)

Features of modulational instability of partially coherent light: Importance of the incoherence spectrum

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(Received 4 September 2003; published 26 February 2004; publisher error corrected 4 March 2004)

It is shown that the properties of the modulational instability of partially coherent waves propagating in a nonlinear Kerr medium depend crucially on the profile of the incoherent field spectrum. Under certain conditions, the incoherence may even enhance, rather than suppress, the instability. In particular, it is found that the range of modulationally unstable wave numbers does not necessarily decrease monotonically with increasing degree of incoherence and that the modulational instability may still exist even when long wavelength perturbations are stable.

DOI: 10.1103/PhysRevE.69.025601 PACS number(s): 42.65.Tg, 05.45.Yv, 42.65.Sf

The modulational instability (MI) of coherent constant amplitude waves in nonlinear Kerr media is one of the most fundamental phenomena resulting from the interplay between nonlinear phase modulation and linear dispersion/diffraction and has attracted much interest over many years [1–3]. Recent advances in the area of nonlinear optics, in particular new results concerning the nonlinear propagation of partially incoherent light and the advent of incoherent solitons [4], have prompted a revisal of this issue during the past decade. The general understanding that emerged from these studies is that the wave intensity threshold for the onset of the MI is increased by the wave incoherence. With this picture in mind, it is remarkable that in a recent investigation of the transverse instability (TI) of solitons [5], it was found that the range of modulationally unstable wave numbers did not monotonously decrease with increasing degree of incoherence. In fact, it first increased until eventually it started to decrease.

Inspired by this result, we consider, in the present work, the problem of the modulational instability of partially coherent waves in more detail and show that the picture is more complicated than previously thought. In order to simplify the analysis and to bring out clearly the new features, the analysis is carried out for the longitudinal modulational instability. We find that the effect of the incoherence on the MI is sensitive to the profile of the incoherent power spectrum. For the often used assumption of a Lorentzian incoherence spectrum, the range of unstable wave numbers does indeed decrease monotonously with increasing degree of incoherence, whereas, e.g., for a Gaussian spectrum, the range first increases and then starts to decrease monotonously. This result agrees well with the unexpected feature observed in [5]. Also, several other subtle effects are shown to be possible. In particular, it is found that the threshold for the MI to be completely quenched is not necessarily associated with the long wavelength limit. Modulations may be stable in this limit, but still be unstable for finite wave numbers. This implies that the threshold for total quench cannot, in a general case, be determined by simplifying the analysis to considering only the long wavelength limit as is done in, e.g., [6,7].

The starting point of our analysis is the normalized nonlinear Schrödinger equation describing the one-dimensional propagation of a partially incoherent wave in a dispersive (or diffractive) nonlinear medium, viz.

$$i(\partial\psi)/(\partial t)+\tfrac{1}{2}\,(\partial^2\psi)/(\partial x^2)+\langle|\psi|^2\rangle\psi=0, \tag{1}$$

where the bracket, $\langle\cdots\rangle$, denotes statistical average [8]. This equation is valid under the assumption that the medium response time is much larger than the characteristic time of the stochastic intensity fluctuations.

The modulational instability of small perturbations of the corresponding steady state solution has been analyzed using different, but equivalent [9], mathematical formalisms. An analysis based on the formalism of the correlation function [5,7,10], or on the Wigner approach [8], results in the dispersion relation

$$\int_{-\infty}^{+\infty}\frac{\rho_0(p-k/2)-\rho_0(p+k/2)}{kp+i\gamma}\,dp=1, \tag{2}$$

with ρ_0 being the Wigner distribution function of the unperturbed cw wave. However, using the transformations: $p+k/2=\theta,\ p-k/2=\theta'$, Eq. (2) can be expressed as

$$\int_{-\infty}^{+\infty}\frac{\rho_0(\theta')d\theta'}{k(\theta'+k/2+i\gamma/k)}-\int_{-\infty}^{+\infty}\frac{\rho_0(\theta)d\theta}{h(\theta-k/2+i\gamma/k)}=1. \tag{3}$$

This is then easily rewritten as

$$k^2\int_{-\infty}^{+\infty}\frac{\rho_0(\theta)d\theta}{k^4/4+(ik\theta+\gamma)^2}=1, \tag{4}$$

which is the same expression as the coherent density approach [11], provided we identify $\rho_0(\theta)=A^2G(\theta)$ with A^2 being the averaged normalized field intensity of the stationary state and $G(\theta)$ being its normalized angular spectrum. Throughout this Rapid Communication we will use both expressions for the dispersion relation interchangeably, since some parts of the analysis are most conveniently handled by one approach, and some by the other.

An explicit analytical solution of the dispersion relation, Eq. (2) or Eq. (4), is possible only for some particular inco-

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herence spectra. Specifically, in the case of the Lorentzian spectrum $G(\theta)=\theta_0/[\pi(\theta^2+\theta_0^2)]$, where θ_0 is the width of the spectrum, it has been shown [8] that the solution can be expressed as

$$\gamma(k,\theta_0)=\gamma_0(k)-k\theta_0 \tag{5}$$

where $\gamma_0(k)$ is the growth rate in the coherent case $[G(\theta)=\delta(\theta)]$ i.e., $\gamma_0(k)=k\sqrt{A^2-k^2/4}$, with $A^2>k^2/4$. This analytical result shows explicitly that the effect of the incoherence, provided it is large enough, is to suppress the MI for any value of the perturbation wave number. However, as will be demonstrated in this Rapid Communication, this result depends crucially on the form of the incoherence spectrum and is not a general feature of the MI of partially incoherent light.

The restricted generality of the result expressed by Eq. (5) can be directly inferred by studying in more detail the properties of the cutoff wave number, k_c, i.e., the value of k at which the growth rate vanishes. According to Eq. (5), valid for the Lorentzian spectrum, k_c is shifted monotonously to the left (decreased) with increasing degree of incoherence, θ_0: $k_c^2=4(A^2-\theta_0^2)$. For the case of a general coherence spectrum, k_c is determined by the following equation:

$$\mathrm{PV}A^2\int_{-\infty}^{+\infty}\frac{G(\theta)d\theta}{k_c^2/4-\theta^2}=1, \tag{6}$$

where PV denotes the Cauchy principal value. When the power spectrum is well localized, in the sense that its rms-width is much smaller than the cutoff wavelength, the contributions from the zeros of the denominator are negligible and the denominator can be expanded to yield

$$\frac{1}{A^2}\approx\int_{-\infty}^{+\infty}\frac{G(\theta)}{k_c^2/4}\left[1+\frac{\theta^2}{k_c^2/4}+\cdots\right]d\theta. \tag{7}$$

Keeping only the first two terms of the expansion one obtains an approximate solution for the effect of partial coherence on the cutoff wave number as follows:

$$k_c^2/4\approx A^2+\langle\theta^2\rangle, \tag{8}$$

where $\langle\theta^2\rangle\equiv\int\theta^2G(\theta)d\theta$. Thus k_c is found to increase for increasing degree of incoherence, which at first sight seems to be in contradiction to the behavior of the previously found exact solution for the Lorentzian spectrum. However, the Lorentz spectrum is not well localized in the sense defined above, since the value $\langle\theta^2\rangle$ does not exist. A first indication of such an incoherence-induced increase of the range of modulationally unstable wave numbers was observed by Torres *et al*. [5] in a numerical study of the transverse instability (TI) of soliton structures using a Gaussian spectral distribution. It should be emphasized though that the effect will occur for transverse as well as for longitudinal modulational instabilities. Although the Lorentz spectrum proper is not well localized, it can easily be made so by considering the bounded Lorentz spectrum, i.e.,

$$G(\theta,\theta_0,\theta_m)=(1/\pi)[\theta_0/(\theta^2+\theta_0^2)]\,CW(\theta_m-|\theta|), \tag{9}$$

where $W(x)=0$ if $x<0$, $W(x)=1$ if $x>0$, θ_m is the boundary of the spectrum and C is a normalization constant. For this spectrum one can explicitly show that, depending on the value of θ_m, the cutoff shift may either increase or decrease with increasing θ_0.

Another perturbative solution of the dispersion equation for the MI is possible in the long wave limit when $k\ll\theta_0$, where θ_0 is the characteristic width of the spectrum $G(\theta)$. In this case one can use a Taylor expansion around p of the numerator in Eq. (2). Introducing $\Gamma=\gamma/k$ we obtain

$$1=-A^2\int_{-\infty}^{+\infty}\frac{G'(p)}{p+i\Gamma}dp-\frac{1}{24}A^2k^2\int_{-\infty}^{+\infty}\frac{G'''(p)}{p+i\Gamma}dp. \tag{10}$$

This simplified approximation [however, without the last term on the right-hand side of Eq. (10)] was used previously [7] to analyze the threshold condition for the suppression of the MI. Actually, assuming $G(p)=G(-p)$ and $\Gamma=\Gamma^*\ll\theta_0$, the first integral can be approximated as

$$\int_{-\infty}^{+\infty}\frac{G'(p)}{p+i\Gamma}dp=-J_1+\pi\Gamma D_1, \tag{11}$$

where

$$J_1=-\int_{-\infty}^{+\infty}\frac{G'}{p}dp,\quad D_1=-\left.\frac{G'}{p}\right|_{p=0}. \tag{12}$$

Within this approximation the dispersion relation becomes

$$\pi D_1\Gamma=J_1-A^{-2}. \tag{13}$$

The equality $J_1=A^{-2}$ can thus be taken as determining the threshold for MI development. While this is true in the limit of vanishing k, it is clear that by taking into account also the next term in the expansion Eq. (10), instead of Eq. (13), one obtains the solution:

$$\pi D_1\Gamma=J_1-A^{-2}-\tfrac{1}{24}k^2J_3, \tag{14}$$

where

$$J_3=\int_{-\infty}^{+\infty}(G'''/p)\,dp. \tag{15}$$

This analysis is valid provided $D_1\neq 0$; in other cases a full solution of Eq. (10) is needed. Performing the calculations one finds that for a Gaussian or a Lorentzian incoherence spectrum, the values of D_1, J_1, and J_3 are all positive quantities. Thus in such cases, Eq. (13) provides a sufficient condition for the suppression of the instability, as found in the works of Anastassiou *et al.* [6] and Soljačić *et al.* [7]. Nevertheless, there may exist spectra for which the factor J_3 is negative, implying there is positive growth of the perturbation for finite k, despite the fact that the solution of Eq. (13) (the long wave limit) gives $\Gamma^2\leq 0$. Consequently, the definition for the suppression of the modulational instability as the threshold value given by the long wavelength limit approximation is not appropriate.

The simplest illustration of the ambiguity of the threshold condition based on the long wavelength limit can be given

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by analyzing a rectangular spectrum profile with $G(p) = 1/2\theta_m$ for p in the interval $-\theta_m < p < \theta_m$, whereas $G(p) = 0$ outside this interval. For this simple spectrum, the integral in Eq. (4) may be evaluated exactly and the following dispersion relation is obtained:

$$\gamma^2 = k^2\left[\frac{k\theta_m}{\tanh(k\theta_m/A^2)} - \frac{k^2}{4} - \theta_m^2\right]. \tag{16}$$

In the limit $k\theta_m \ll A^2$, the dispersion relation can be approximated to read

$$\gamma^2 \approx k^2\left[A^2 - \theta_m^2 - k^2\left(\frac{1}{4} - \frac{\theta_m^2}{3A^2}\right)\right]. \tag{17}$$

The shift of the cutoff wave number k_c is given approximately by $k_c^2 \approx 4(A^2 + \theta_m^2/3)$, provided $\theta_m \ll A$, which is in full agreement with Eq. (8). However, the rectangular spectrum exhibits one more unexpected and very important feature: It is evident that even if $\gamma^2 < 0$ in the limit as $k \to 0$, γ^2 may become positive for finite k. In particular, when $k = 2\theta_m$, the growth rate of the perturbation is positive, $\gamma^2 > 0$, independently of the spectrum width θ_m, in fact

$$\gamma^2 = \frac{16\theta_m^4}{\exp(4\theta_m^2/A^2) - 1} > 0. \tag{18}$$

Therefore even a very high degree of incoherence does not completely suppress the modulational instability when the spectrum is rectangular; cf. Fig. 1.

Even when $\theta_m \gg A$ and the MI is strongly suppressed within the long wave range, there still exists a "resonance" region of instability for wave numbers k around $2\theta_m$ given by $|k - 2\theta_m| < 4\theta_m \exp(-\theta_m^2/A^2)$. It is interesting to note that a similar phenomenon was found in [12] in an investigation of the effect of a nonlocal nonlinearity on the MI in a focusing Kerr medium. In particular, it was shown for a rectangular response function that the instability growth rate first decreased for increasing width of the response function, but ultimately for large widths new modulationally unstable bands appeared at finite wave numbers.

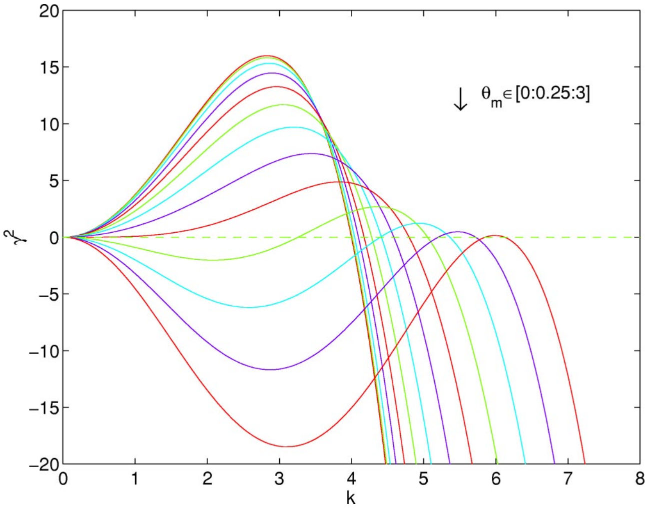

FIG. 1. The effect of increasing rectangular spectrum width θ_m on the MI. The parameter θ_m runs from $\theta_m = 0$ (the top most curve) in increments of 0.25 to $\theta_m = 3$ (the bottommost curve). As can be seen, the instability is never completely suppressed.

Since a rectangular profile is a very ideal and a bit artificial form of the spectrum, we consider another example of a well localized spectrum in the form of a modified Lorentzian, which exhibits similar properties to those of a rectangular profile, $G(\theta) = (\sqrt{2}/\pi)[\theta_0^3/(\theta^4 + \theta_0^4)]$. Even in this case the integral of Eq. (4) can be calculated in explicit form to yield

$$\int_{-\infty}^{+\infty} \frac{G(\theta)d\theta}{(\gamma + ik\theta)^2 + (k^2/2)^2} = \frac{k^4/4 - \dfrac{k^2\theta_0^2}{2} + \left(\gamma + \dfrac{k\theta_0}{\sqrt{2}}\right)^2 + 2\left(\gamma + \dfrac{k\theta_0}{\sqrt{2}}\right)\dfrac{k\theta_0}{\sqrt{2}}}{\left[\left(\gamma + \dfrac{k\theta_0}{\sqrt{2}}\right)^2 + \left(\dfrac{k\theta_0}{\sqrt{2}} + \dfrac{k^2}{2}\right)^2\right]\left[\left(\gamma + \dfrac{k\theta_0}{\sqrt{2}}\right)^2 + \left(\dfrac{k\theta_0}{\sqrt{2}} - \dfrac{k^2}{2}\right)^2\right]}. \tag{19}$$

Using this expression in the cutoff condition, $\gamma^2 = 0$, we obtain the following result for the cutoff wave number:

$$k_c^2 = 2A^2\left\{1 \pm \sqrt{1 + 4\left(\frac{\theta_0}{A}\right)^2\left[1 - \left(\frac{\theta_0}{A}\right)^2\right]}\right\}. \tag{20}$$

When $1 < (\theta_0/A)^2 < (1+\sqrt{2})/2$, the equation has two positive roots implying that the MI is not completely suppressed provided this condition is fulfilled. On the other hand, $\gamma^2 > 0$ only within a limited range of wave numbers, viz.,

$$\left|\frac{k^2}{2A^2} - 1\right| < \sqrt{1 + 4\left(\frac{\theta_0}{A}\right)^2\left[1 - \left(\frac{\theta_0}{A}\right)^2\right]}. \tag{21}$$

In order to further illustrate the subtlety of the interplay between the partial incoherence and the instability drive, we consider a multicarrier case where the field consists of many mutually incoherent, but individually partially coherent waves with a spectrum given by $G(\theta) = \Sigma_n G_n(\theta - \theta_n)$. The dispersion relation then becomes

$$A^2k^2\sum_n \int_{-\infty}^{+\infty} \frac{G_n(\theta-\theta_n)d\theta}{(\gamma+ik\theta)^2+(k^2/2)^2}=1. \quad (22)$$

For simplicity we consider the particular case of equally separated carriers, i.e., $\theta_n=\alpha n$ where n is an integer, with each carrier having a Lorentzian phase spectrum of the same width

$$G_n(\theta-\theta_n)=\frac{g(n)\theta_0}{\pi[(\theta-\alpha n)^2+\theta_0^2]}. \quad (23)$$

The dispersion relation, Eq. (22), can then be written as

$$A^2k^2\sum_n \frac{g(n)}{(\Gamma+ik\alpha n)^2+k^4/4}=1, \quad (24)$$

where $\Gamma=\gamma+k\theta_0$. For a symmetric spectrum, i.e., when $g(k)=g(-k)$, this dispersion relation can be rewritten in terms of real functions:

$$A^2\sum_n g(n)\frac{X+(a+n\alpha)(a-n\alpha)}{[X+(a+n\alpha)^2][X+(a-n\alpha)^2]}=1, \quad (25)$$

where $X=\Gamma^2/k^2$, $a=k/2$, and the coefficients $g(n)$ are normalized to unity, $\Sigma_n g(n)=1$. When X has a small positive value, the sum in Eq. (25) has multiple resonant values at $a_n=\alpha n$, $n\neq 0$. Consequently, one should expect the existence of small positive roots X of Eq. (25) in the vicinity of those resonances, independently of the structure of the spectrum envelope $g(n)$. That this indeed is the case is illustrated in Fig. 2, which shows the result of a numerical evaluation of the sum for particular parameter values.

On the other hand, outside of these resonant bands, the summation can be transformed into an integration over n provided the spectrum $g(n)$ is dense enough, i.e., $\alpha \ll A$, $|g(n+1)-g(n)|\ll g(n)$;

$$A^2\int_{-\infty}^{+\infty} g(n)\frac{X+(a+n\alpha)(a-n\alpha)}{[X+(a+n\alpha)^2][X+(a-n\alpha)^2]}dn=1. \quad (26)$$

This equation coincides with Eq. (4) if $g(n)dn$ is replaced by $G(\theta)d\theta$ and $n\alpha$ is changed to θ. It corresponds to the MI of a continuous spectrum which coincides with the envelope of the actual spectrum. For example, when $g(n)=(1/\pi)\times[n_0/(n^2+n_0^2)]$, where $n_0\gg 1$, Eq. (26) is reduced to the following well-known expression: $(\Gamma+k\alpha n_0)^2=k^2(A^2-k^2/4)$. Thus, within this approximation, the MI is suppressed for all wave numbers provided $\alpha n_0>A$. However, as shown above, this is correct only outside the resonance bands, i.e., the localized regions around each $2\alpha n$ $(n\neq 0)$, where $\Gamma^2>0$, independently of the width of the envelope (n_0). To conclude, we have investigated the role of the incoherent spectrum profile on the properties of the modulational instability. The gain curve of the instability may smoothly shrink in amplitude and cutoff wave number with increasing degree of incoherence, as is the case for a Lorentzian profile. However, it may also initially expand into the wave numbers which are stable in the coherent regime as is the case for a Gaussian profile of the spectrum. When the spectrum is rectangular, we have shown that under the special resonance condition $k=2\theta_m$, the MI cannot be suppressed completely no matter how strong the degree of incoherence. Using the modified Lorentzian profile we demonstrate that the long wavelength threshold definition at which the MI is suppressed due to partial incoherence is in fact profile dependent. For this case, islands of positive growth rate emerge at higher wave numbers. Lastly, our analysis of the modulational instability in the case of multicarrier operation demonstrates new additional structure in the gain curve, where besides the main lobe, there also exist smaller peaks surrounding the discrete carrier phases.

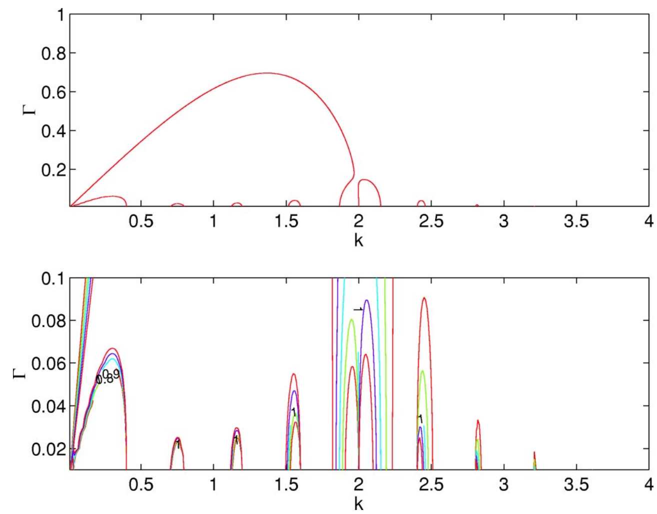

FIG. 2. Numerical solution of Eq. (25) for multicarrier operation. The number of carriers is 10, $\alpha=0.2$, and $n_0=0.8$. Top: Main growth curve together with the small instability islands stemming from the separate carriers. Bottom: An enlargement of the instability curve. Notice the resonant peaks occurring beyond the region of the main curve.

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